Tutorial Week 9

1. Let $P(n)$ be the statment $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ where n is greater than or equal to 1.
   a) What is the statement $P(1)$?
   b) Show that $P(1)$ is true.
   c) What is the inductive hypothesis?
   d) What do you need to prove the inductive step?
   e) Complete the inductive step.
   f) Explain why these steps show how that this equality is true whenever n is an integer greater than or equal to 1.

2. Let $P(n)$ be the statment $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$, where n is greater than 1.
   a) What is the base case?
   b) Show that the base case is true.
   c) What is the inductive hypothesis?
   d) What do you need to prove the inductive step?
   e) Complete the inductive step.
   f) Explain why these steps show how that this inequality is true whenever n is an integer greater than 1.

3. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

4. Let $P(n)$ be the statement $1^2 - 2^2 + 3^2 \cdots + (-1)^{n-1}n^2 = \frac{(-1)^{n-1}n(n+1)}{2}$ Prove this statement to be true.

5. What is wrong with the following proof?
   We want to prove that a warren of rabbits will all be the same colour.
   For the basis of induction pick any one rabbit. By default, the rabbit is of the same colour with itself, and hence the entire warren of 1 is the same colour.
   For the inductive step, assume that any set of k rabbits have been proven monochromatic.
   Choose a warren of $n+1$ rabbits. Temporarily remove one rabbit from the set, leaving a set, denoted A, of n rabbits, and this one lone rabbit can be set B. By the induction hypothesis, these n rabbits are of the same color. Put the 'unused' rabbit from set B back into the set and remove another one, creating a set A_2 and B_2. As before, all rabbits in set A_2 are of the same colour, and of course B_2 is one rabbit, the same colour.
   All the rabbits on hand (except for the special two) belong to both sets A_2 and B_2, implying that the rabbits in both sets are of the same colour. But the union $A \cup B$ contains all $n + 1$ rabbits which thus we have shown are all of the same colour.

6. In a magical land of dragons and dinosaurs where Patrice lives, there are only 3-cent and 7-cent stamps. The post-office charges 1-cent per mile they have to ride their magical ostriches to deliver letters. Because of the cost of using magical ostriches, they will only ride their ostriches if the delivery is further than 11 miles.
   Prove Patrice can send any letter of n cents, so long as $n > 11$ to his/her relatives and doesn't have to brave the reptilian dangers him/herself.

7. You are given a rectangular piece of graphing paper, that contains a total of n squares (for instance, you might have a paper with 3 rows of 2 squares ($n = 6$) or a paper with 17 rows of 13 squares ($n = 221$). Your goal is to separate all the small squares from one another (so you end up with n single squares). The only operation you are allowed to perform is pick one of the pieces of paper you currently have (these will be parts of the piece you started with) and cut all the way along one of the lines (see below of an example with 6 squares). Use (strong) mathematical induction to prove you will always need to make exactly $(n - 1)$ cuts, regardless of the sequence of cuts.