Midterm 2 Review

March 19th, 2012

1 Topics

i. Predicate Logic

ii. Formal Proofs & Techniques

iii. Deterministic Finite State Automata

iv. Flipflops, Circuit Memory

v. Induction

2 Examples

I Use the following to answer each question in two equivalent different ways, one existential, one universal:

- $\text{Dog}(x) : x$ is a dog
- $\text{Green}(x) : x$ is green-coloured
- $\text{LargerThan}(x, y) : x$ is larger than y
- $\text{SameSize}(x, y) : x$ is the same size as y
- A, the set of all animals

(a) There is an animal that is green.

$\exists x \in A, \text{Green}(x)$
$\sim\forall x \in A, \sim\text{Green}(x)$

(b) No dogs are green.

$\forall x \in A, \text{Dog}(x) \rightarrow \sim\text{Green}(x)$
$\sim\exists x \in A, \text{Dog}(x) \wedge \text{Green}(x)$

(c) There is a non-dog that is larger than a dog.

$\exists x \in A, \sim\text{Dog}(x) \wedge (\exists y \in A, \text{Dog}(y) \wedge \text{LargerThan}(x, y))$
$\sim\forall x \in A, \sim\text{Dog}(x) \rightarrow (\forall y \in A, \text{Dog}(y) \rightarrow \sim\text{LargerThan}(x, y))$

II Convert these proofs to logical statements and outline how you would approach the proof:

(a) Prove that if a and b are real numbers, $a \neq 0$, then there is a unique real number r s.t. $ar + b = 0$

Direct proof. Assume true, and use the second fact to move into a fraction. Then try finding values of r.
Turns out it doesn't work but...

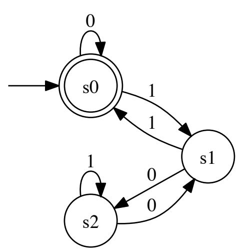
(b) Prove that if m is a perfect square, $m+2$ is not a perfect square

Attempt direct proof. i.e.
Assume m is a perfect square, $m+2$, e.g. $m = a^2$ for some $a \in \mathbb{Z}$
Work with $(a+1)$ to show that we have to be at least $m+3$ to work.

(c) Prove that if x^3 is irrational, x is irrational.

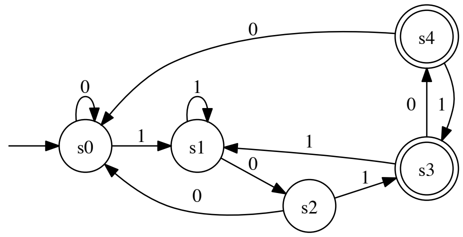
Use contrapositive.
Assume x is not irrational, thus x is rational.
This means $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0$.
Get the form x^3 =some fraction.
Thus we can conclude by contrapositive that if x^3 is irrational, so must x be.

III Describe the languages the following DFA's accept:



(a)

Binary numbers divisible by 3.



(b)

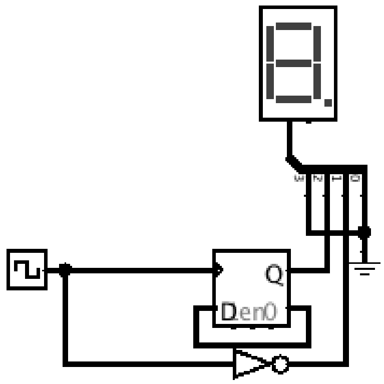
Numbers that end in 101 or 1010.

IV Consider the following Circuit questions:

(a) How does one get an 'initial' value for a flip-flop? How do we prevent using this value each clock-tick?

Feed in the clock value with an exclusive or.

(b) Design a circuit that counts even numbers up to 6 (i.e. 2, 4, 6, 0)



V Induction Questions:

(a) Prove $P(n) \leftrightarrow 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

Base case: $n = 0$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$
$$1 = 2^{(0+1)-1}$$
$$1 = 2 - 1$$
$$1 = 0$$

Inductive hypothesis: Assume this holds for $n = k$

We have

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

and we will now show this is true for $n = k + 1$

$$\sum_{i=0}^{k+1} 2^i + 2^{k+2} = 2^{(k+1)+1} - 1$$
$$\sum_{i=0}^{k+1} 2^i + 2^{k+2} = 2^{k+2} - 1$$

starting from $n = k$

$$\begin{aligned} \text{Right side of equation} &= 2^{k+1} - 1 \\ &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

And thus our right hand side = left hand side.

Thus we have shown this statement applies for $n = 1$ and proves it for $n = k \rightarrow n = k + 1$.
It then follows that this holds for $n = 1, 2, 3, \ldots$

(b) $x, y, n \in \mathbb{N}$
If $x < y$, then $x^n < y^n$

Base case: $n = 1$
If $x < y$ then $x^1 < y^1$
This statment is correct, thus we move to the inductive hypothesis.

Inductive hypothesis: Assume this statement now holds for $n = k$
If $x < y$ then $x^k < y^k$
Since $0 < a < b$ and $0 < x < y$, we can state $ax < bx$ and thus $bx < by$
By knowing this, we can state: if $x^k < y^k$ then, by above, $(x^k)x < (y^k)y$
This is $x^{k+1} < y^{k+1}$
Since $x < y$ this statement must hold $\forall k$.

Thus we can state this holds $\forall n$ as long as $x, y, n \in \mathbb{N}$

3 Additional Relevant Questions

The following questions I chose as being particularly useful in what was missing from this sheet. Note I have not included some topics, but this is not to say they are not on the midterm ***Especially*** don't forget predicates, quantifiers etc.

- 2006/7 Winter Term 2 Good proof question: $(p \oplus q) \vee (p \wedge q) \vee r \equiv p \vee (q \vee r)$
- 2009W1 Part 2, Part 5-2. Unusual propositional logic question, more difficult proof
- 2011S Question 3 is kindof fun :) Understanding some made-up laws
- 2010W2 Q3, Q6 Propositional translation, Circuits
- 2007/8W1 Question 4 A harder midterm question about number representation and circuits